Quantum Expression of Classical Chaos

Sudhir R. Jain[(1)], Benoît Grémaud[(2)], Avinash Khare[(3)]
[(1)] Theoretical Physics Division
Bhabha Atomic Research Centre
Mumbai 400094, India

[(2)] Laboratoire Kastler Brossel
Université Pierre et Marie Curie
T12 E1, 4 place Jussieu
75252 Paris Cedex 05, France

[(3)] Institute of Physics, Sachivalaya Marg
Bhubaneswar 751005, India

Abstract

Even as we understand for long that the world is quantal and buried in it is classical dynamics which is chaotic, finding eigenfunctions analytically from the the Schrödinger equation has turned out to be a near-impossibility. Here, we discover a class of chaotic quantum systems for which we obtain some analytically exact eigenfunctions in closed form. This paves way to an exact classical and quantum mechanical treatment of chaotic systems. Furthermore, we bring out connections, underlying the discovery, between different areas of physics and mathematics related to universality in fluctuations observed in a wide variety of complex quantum systems.

1 Introduction

World around us is rather complex where nonlinear phenomena abound. Nonlinearities give birth to chaos and make it impossible to predict long-time dynamics *(1,2)*. Chaotic behaviour is characterized by the existence of positive Lyapunov exponents, which determine the rate of exponential separation of very close trajectories in the phase space of the system. Upon casting the chaotic systems in a quantum mechanical framework, impressions of chaos are found in variety of statistical properties of energy levels and eigenfunctions *(3-5)*. The fluctuation properties of energy level sequences of chaotic quantum systems agree very well with the results in random matrix theory (RMT) *(6)*. These extensive studies have given place to quantum chaology or what we call 'statistical quantum chaos'.

Eigenfunctions and eigenvalues contain all the information about a time-independent quantum system. Chaotic eigenfunctions have been studied in detail and there are two main ideas around which the general understanding has evolved. According to Berry *(7)*, for chaotic systems, eigenfunctions corresponding to excited states are conjectured to be well-represented as a random superposition of plane waves. This conjecture has a lot of numerical support on one hand, and, is connected to statistical mechanics on the other *(8,9)*. The other idea ensues from Heller's discovery *(10,11)* of scarring of eigenfunctions by periodic orbits, where the probability density is considerably enhanced on the periodic orbit in configuration or phase space. This discovery has helped in understanding how classical periodic orbits form the underlying fabric for quantum states, and it helps in appreciating the beautiful nodal patterns and contour plots of chaotic eigenfunctions. Nevertheless, an analytical expression isn't obtained, and, Berry's conjecture does not help in seeing how coefficients in the superposition arrange to give patterns.

The objective we set out for ourselves is to get analytically exact expressions for chaotic eigenfunctions in closed form. As we shall see, the resulting aesthetic beauty is due to some 'magic' connections *(12)* involving RMT and its relation to other topics in physics and mathematics, including complex quantum systems, Riemann zeta function, exactly solvable many-body problems, partial differential equations and the Riemann-Hilbert problem *(13)*. These different areas are related to each other by the statistical properties of sequences characterizing them, like energy levels of nuclei and chaotic quantum systems, zeros of the Riemann zeta function, and eigenvalues of transfer matrices of disordered conductors. Moreover, joint probability distribution function (JPDF) of eigenvalues of random matrices is given by a statistical mechanics problem involving particles with Coulomb-like interactions *(14)*. The JPDF gives the probability with which eigenvalues $\{E_i\}$ are found in intervals $[E_i, E_i + dE_i]$; all the correlation functions giving various physical quantities follow from this. These problems belong to the general class of exactly solvable statistical mechanics models. In turn, these models can be converted to problems in Hamiltonian dynamics where one can study the classical and quantal aspects.

In this paper, we extend these connections to another class of random matrix models developed to explain intermediate statistics found in pseudointegrable systems, the Anderson model in three dimensions at the metal-insulator transition point and in certain problems in atomic physics *(15-18)*. The sequences involved in these systems exhibit a behaviour intermediate to regular and chaotic. Once again, the eigenvalue distribution of such random matrices is related to an exactly solvable statistical mechanics model where the interaction is a screened Coulomb-like along with an additional three-body term *(19)*. In this article, the abovementioned model is mapped into a problem in Hamiltonian dynamics in d dimensions which is shown to dis-

play chaos. Thus, the exactly solved many-body problem is non-integrable in the sense that there are lesser number of constants of the motion than the degrees of freedom. Quantum mechanically, some chaotic eigenfunctions are then found analytically even though the underlying classical dynamics at those energies is chaotic. This is illustrated through Sections 2 to 4, the ensuing general perspective is given in Section 5.

It is worth noting that the existence of Bose-Einstein condensation in a related one-dimensional many-body problem at zero temperature is proved recently *(20)*. To prove the existence of a Bose-Einstein condensate in one dimension at non-zero temperatures, we need the excited states. Thus, the development presented here is of importance to the general theory of quantum phase transitions.

For quantum cat maps (toral automorphisms), analytical form for the eigenfunctions is known *(21,22)*, where the solutions were possible because the semiclassical studies turned out to be exact. These are very interesting results but the methods are so specialized that they do not throw much light on other related problems. In Section 5, we discuss the possible advances emerging from the discussions that follow in the context of general relevance of the results and the theme of this article.

2 From Many-Body Problem to One-Body Problem

In 1999, a many-body problem in one dimension was discovered *(19,23)* where the nearest neighbours interact via a repulsive interaction which is inverse- square in distance between the particles and an attractive three-body interaction, also inverse-square in position coordinates of the particles. The N-particle problem on a circle has the Hamiltonian :

$$\begin{aligned} H &= \sum_{i=1}^{N} \frac{p_i^2}{2m} + g\frac{\pi^2}{L^2}\sum_{i=1}^{N} \sin^{-2}\left[\frac{\pi}{L}(x_i - x_{i+1})\right] \\ &- G\frac{\pi^2}{L^2}\sum_{i=1}^{N} \cot\left[\frac{\pi}{L}(x_{i-1} - x_i)\right]\cot\left[\frac{\pi}{L}(x_i - x_{i+1})\right] \ , \end{aligned} \tag{1}$$

with $x_j = x_{N+j}$. In the rest of this article, we will take the mass as unity and the circumference $L = \pi$, for notational simplicity. The potential is singular whenever $x_{j+1} = x_j + n\pi$ (n being an integer) with an inverse-square singularity. This leads to disconnected domains where the wavefunctions are zero at singular (hyper-)planes in the quantum problem. We choose the domain in which the particles are ordered as $x_1 \le x_2 \le ... \le x_1 + \pi$. The center of mass (CM) motion can be separated in this case by writing the amplitudes of the motion of particles around the CM, $X = \frac{1}{N}\sum_{i=1}^{N} x_i$ in terms of normal mode coordinates *(24)*. Thus, we can write the positions as

$x_j = X + y_j$, and y_j's as

$$
\begin{aligned}
y_j &= -\frac{\pi}{2} + \left(j - \frac{1}{2}\right)\frac{\pi}{N} + \frac{1}{\sqrt{N}} q_M \cos(\pi j) \\
&+ \sqrt{\frac{2}{N}} \sum_{n=1}^{M-1} \left[q_n \cos\left(\frac{2\pi n j}{N}\right) - q_{-n} \sin\left(\frac{2\pi n j}{N}\right)\right], \qquad (2)
\end{aligned}
$$

for even N ($M = N/2$); and for odd N, without the last term on the right hand side and summation going upto $n = M$. The quantum Hamiltonian operator transforms to

$$
\begin{aligned}
H &= H_{CM}(X) + H_{\text{billiard}}(\{q_n\}) \\
&= -\frac{1}{2N}\frac{\partial^2}{\partial X^2} - \frac{1}{2}\sum_{n}^{M} \frac{\partial^2}{\partial q_n^2} + \sum_{j=1}^{N} W[y_j(\{q_n\})], \qquad (3)
\end{aligned}
$$

where $W[y_j(\{q_n\})]$ is the potential term in (1). Note that $n = \pm 1, \pm 2, ..., \pm M$ for odd N while $n = \pm 1, \pm 2, ..., \pm(M-1), M$ for even N. Thus, H_{billiard} represents a single particle in an $(N-1)$- dimensional domain bounded by the (hyper-)planes where potential becomes singular forcing all the wave functions to be zero on the boundaries of the domain. This gives us a class of billiards as a function of β (defined through $G = \beta^2$ and $g = \beta(\beta-1)$) and from dimensions two to $(N-1)$. A similar connection was realized by Rey and Choquard [24] when they considered the Calogero-Sutherland-Moser (CSM) system and showed that the N-body problem in this case is mapped to an integrable billiard problem in $N-1$ dimensions. It is interesting to note that the CSM and our model coincide for three particles as nearest neighbours are all the neighbours and the cotangent term is unity owing to a trigonometric identity. Unlike this, and most interestingly, our many-body problem leads to non-integrable billiards for particles greater than three and hence, billiards of dimension greater than two. We now turn to show that for $N > 3$, our billiard models are, in fact, chaotic. This also implies non-integrability of the many-body problem.

3 Family of Classical Billiards

For the sake of concreteness, we concentrate our discussion on $N = 4$, leading to a three-dimensional billiard. Equations of motion are singular each time a collision occurs (involving 2, 3 or 4 particles), but two-particle collisions are forbidden by energy conservation, giving rise to smooth integration using standard Runge-Kutta method, until one reaches a multiple collision *(25)*. Preliminary studies show that regularization of the classical motion near these collisions could be done, but its exact implementation has not been achieved yet. Nevertheless, we believe that this problem does not alter results presented in this section. For each trajectory, we have also computed the

associated monodromy matrix, M (whose symplectic structure is used as a relevant test), allowing us to extract evidence of chaotic behaviour in our system. For this, we have computed the "Liapunov exponents", $\tilde{\lambda}(T) = \ln(|M(T)\cdot \mathbf{e}_0|)$ using three different vectors : $\mathbf{e}_0^{\parallel}$, $\mathbf{e}_0^{\perp}$ and $\mathbf{e}_0^{\mathrm{r}}$, which are, respectively, unit vector parallel to the flow at an initial time, unit vector perpendicular to the energy shell at an initial time and a "random" unit vector, namely $1/\sqrt{6}(1,1,1,1,1,1)$. $\tilde{\lambda}^{\parallel}(T)$, being equal to $\ln|\dot{\mathbf{X}}(T)|/|\dot{\mathbf{X}}(0)|$, is bounded entailing thereby a vanishing Liapunov exponent, providing a reference scale for further numerical estimation of non-vanishing Liapunov exponent. For $\beta = 2$, at classical energy equal to the quantum ground state energy $\epsilon_0 = 4\beta^2$ (see Eq. (6)), and initial conditions,

$$\begin{pmatrix} q_1(0) \\ q_{-1}(0) \\ q_2(0) \end{pmatrix} = \begin{pmatrix} 0.1000 \\ 0.2000 \\ 0.0000 \end{pmatrix} \quad \begin{pmatrix} p_1(0) \\ p_{-1}(0) \\ p_2(0) \end{pmatrix} = \begin{pmatrix} 1.0000 \\ 2.0000 \\ 5.1844 \end{pmatrix} \tag{4}$$

results on Liapunov exponents are plotted in Fig. **??**. As expected, behavior of $\tilde{\lambda}^{\parallel}(T)$ is substantially different from those of $\tilde{\lambda}^{\perp}(T)$ and $\tilde{\lambda}^{\mathrm{r}}(T)$, emphasizing thus the presence of hard chaos in the system, with Liapunov exponent, $\lambda \sim 6$ *(26)*.

A more appealing evidence of hard chaos is obtained on plotting Poincaré surface of sections (PSOS). For a generic 3D time-independent system, their dimensionality (4D) makes them quite useless for visualising. Fortunately, in our case, symmetry properties of the Hamiltonian, viz., invariance under $q_1 \leftrightarrow q_{-1}$ exchange allows us to consider the reduced phase space made of trajectories for which $q_1 = q_{-1}$ and $p_1 = p_{-1}$ at any time, leading thus to an effective 2D system. Fig. **??** depicts, for $\beta = 2$, the reduced PSOS defined by $q_2 = 0$, at different energies : $E_a = \epsilon_0 = 16$ (ground state energy, see Eq. (6)), $E_b = \epsilon_1 = 25.5$ (see Eq. (10)), $E_c = \epsilon_N = 36$ (see Eq. (12)) and $E_d = 100$. The empty areas appearing on all these plots correspond to trajectories having a four body collision in their past, this has been checked by propagating the corresponding initial conditions backward in time *(27)*. Nevertheless the system appears to be fully chaotic in the reduced phase space, which is emphasized by the fact that all periodic orbits of the reduced dynamics are unstable (i.e., non-trivial eigenvalues of the monodromy matrix are not on the unit circle). Actually these orbits are also unstable when considered in the full phase space. Moreover, there are also unstable periodic orbits not belonging to the reduced phase space. Even more, we find unstable periodic orbits for which the four non-trivial eigenvalues of the monodromy matrix form a quadruplet $(\Lambda, \Lambda^*, \Lambda^{-1}, \Lambda^{-1*})$, Λ being complex. For example, for $\beta = 2$, there is a periodic orbit of length 1.1083386 with initial conditions (classical energy is ϵ_0) :

$$\begin{pmatrix} q_1(0) \\ q_{-1}(0) \\ q_2(0) \end{pmatrix} = \begin{pmatrix} -0.228018 \\ -0.261062 \\ 0.000000 \end{pmatrix} \quad \begin{pmatrix} p_1(0) \\ p_{-1}(0) \\ p_2(0) \end{pmatrix} = \begin{pmatrix} 2.06029 \\ 0.737421 \\ 5.07910 \end{pmatrix} \tag{5}$$

for which we find $\Lambda = -367.64192 + \iota 64.633456$, this is shown in Fig. ??.

$\beta = 2$ is not a special value, we have checked that all results presented in this section hold for other values $\beta > 1$, with eventually apparition of a mixed dynamics for large β. A complete study of classical dynamics has to be done, especially a systematic search for periodic orbits and their properties (variations with both β and energy, bifurcations, and so on).

4 Family of Quantum Billiards

We shall now use the known exact energy eigenstates *(28)* of the newly discovered N-body Hamiltonian (1) and obtain few energy eigenvalues and eigenfunctions of the corresponding $(N-1)$-dimensional billiard. The billiard eigenfunctions are obtained by eliminating the center-of-mass dependence while the eigenvalues are determined by subtracting off the center-of-mass energy.

Let E_k, ψ_k denote the energy eigenvalues and eigenfunctions of the N-body problem (1) with periodic boundary conditions, i.e. $H\psi_k = E_k\psi_k$. Then the exact ground state is given by *(19,23)*

$$\psi_0 = \prod_j^N \mid \sin(x_j - x_{j+1}) \mid^\beta \quad , \quad E_0 = N\beta^2 \ , \tag{6}$$

provided g, G are related by $g = \beta(\beta-1), G = \beta^2$.

As mentioned in the Introduction, ψ_0^2 exactly gives the analytical form of the JPDF of eigenvalues of random matrix ensemble relevant for intermediate statistics *(15,18,19)* observed in plane polygonal billiards *(15,29)*, Aharonov-Bohm billiards *(17)*, Anderson model in three dimensions *(16)*, and so on.

In addition to the ground state, a few of the excited energy eigenstates have also been obtained in this case *(28)* and are given by ($\psi_k = \psi_0\phi_k$)

$$\begin{aligned}
\phi_1 &= e_1 \ , \quad E_1 = E_0 + 2 + 4\beta \ , \\
\phi_{N-1} &= e_{N-1} \ , \quad E_{N-1} = E_0 + 2N - 2 + 4\beta \ , \\
\phi_N &= e_1 e_N - \frac{N}{1+2\beta} e_N \ , \quad E_N = E_0 + 2N + 4 + 8\beta \ .
\end{aligned} \tag{7}$$

Here e_j, $(j = 1, 2, ..., N)$ is an elementary symmetric function of order j in the variable z_j. For example $e_2 = z_1z_2 + z_1z_3 + ...$ (containing $N(N-1)/2$ terms) and $z_j = \exp(2\iota x_j)$. Note that ϕ_k is an eigenfunction of the momentum operator with eigenvalue k *(30)*. Following the treatment in *(24)* for the CSM billiard, it is easily shown that if the billiard Hamiltonian satisfies the eigenvalue equation $H_B\chi_k = \epsilon_k\chi_k$, then the eigenvalues ϵ_k and the eigenfunctions χ_k are related to E_k, ψ_k by $\chi_k = \exp(-2\iota kX)\psi_k$ with

$\epsilon_k = E_k - \frac{2k}{N}$. Finally, we then obtain the following excited eigenvalues and eigenfunctions for the $(N-1)$-billiard

$$\chi_{1c} = \psi_0 \left[\cos \frac{2}{N}([N-1]x_1 - x_2 - ... - x_N) + \text{cyclic permutations} \right], \quad (8)$$

$$\chi_{1s} = \psi_0 \left[\sin \frac{2}{N}([N-1]x_1 - x_2 - ... - x_N) + \text{cyclic permutations} \right], \quad (9)$$

$$\epsilon_{1c} = \epsilon_{1s} = E_0 + 4\beta + 2 - \frac{2}{N} \ , \quad (10)$$

$$\chi_N = \psi_0 \sum_{i<j=1}^{N} \cos 2(x_i - x_j) + \frac{N\beta}{1+2\beta} \ , \quad (11)$$

$$\epsilon_N = E_0 + 8\beta + 4 \ , \quad (12)$$

where E_0 is as given by Eq. (6). Note that even though χ_N is N-dependent, the energy difference $\epsilon_N - E_0$ is N-independent.

These expressions conclude our illustration of the connections between different areas of research in physics and mathematics, as noted in the Introduction and by Kadanoff *(12)*. The classical dynamics at energy equal to the energy eigenvalues of the eigenfunctions is chaotic, yet the eigenfunctions are not random superposition of plane waves. However, since β can assume any value, the functions could be quite complicated, yet comprehensible.

5 Summary

We underline a fundamental theme which is at the heart of our findings. Let us first present two scenarios - *Scenario 1:* From pseudointegrable systems to a class of fully chaotic systems, and, *Scenario 2:* From fully chaotic systems to a class of integrable systems.

Scenario 1

Let us begin with classically pseudo-integrable billiards which are neither chaotic (zero Kolmogorov-Sinai (KS) entropy) nor integrable. On quantization, we get the energy levels and eigenfunctions. The local statistical properties of energy levels agree well with the Short-Range Dyson Model (SRDM). The JPDF of eigenvalues is such that its analytical form coincides with the square of the ground-state wavefunction of the many-body problem (Cf. Section 2). As shown in Section 2, this many-body problem is mapped to a family of quantum billiards. The classical analogues of these quantum billiards are fully chaotic.

Scenario 2

We begin with classically chaotic systems and note that on quantization, the local fluctuation properties agree well with those of canonical RMT. This is very well-studied. The JPDF of eigenvalues of the random matrix ensembles is such that its analytical form coincides with the square of the ground-state wavefunction of the CSM. In the CSM, there are N particles on a circle and each pair interacts through an inverse-square potential. This is a completely integrable system. Since the ground-state is crystalline, here again the mapping to billiards in $(N-1)$ dimensions is possible. However, in this case, the billiards are fully integrable for any N.

Thus, we see that the two random matrix models - canonical and the SRDM - both have two arms, one arm leads to chaos ($h_{KS} > 0$) and the other to order ($h_{KS} = 0$). The arm which leads to analytical results for chaotic systems belongs to the SRDM and the associated many-body problem due to Jain and Khare *(19)*. Thus, we suggest that for quantizing chaotic systems, the route through many-body problems may be more worthwhile. Of course, it would require a great deal of ingenuity to construct such many-body problems. However, the recent work of Glashow and Mittag *(31)* shows that it is possible for any triangular billiard, where the authors generalise an old connection of Onsager and Sinai *(32)*.

There is another important future direction which suggests itself from the Scenarios presented above. The quantum systems that we usually deal (billiards, for instance) with, have an infinite-dimensional Hilbert space. Thus the Hamiltonian matrix is infinite-dimensional. This means that the random matrices also should be of an infinite order, which manifests as universality in the scaling limit. This would lead to many-body problem with infinite degrees of freedom - known as fields or classical fluids governed by partial differential equations. An instance of relevance here is the Korteweg-de Vries equation associated with the CSM. It is only the truncations and projections of these field solutions and the vast variety of solutions admitted by the partial differential equations that manifest themselves in the standard parlance and certainly in numerical works. As discussed by Deift *(13)*, the solvability of certain partial differential equations demands the solution of the Riemann-Hilbert problem which, in turn, brings in matrix theory. We believe that herein underlie some deep secrets of physical theories.

Finally, a concluding remark on quantum theory of chaotic systems. We have seen above that low-lying eigenfunctions built on classical chaos - termed as 'Chaotic States' - are not random superpositions of plane waves. Thus, Berry's conjecture would be expectedly true for highly excited chaotic states, and not for chaotic states in general. We believe that chaos in quantum wavefunction, even at low energies, would show up as the system evolves. The time-dependent wavefunction, written as a superposition of eigenfunc-

tions, has coefficients displaying chaos. Thus, eigenfunctions form an invariant set in quantum theory, much in the same way as periodic orbits and fixed points do in classical theory. The most important evidence is shown by Heller's discovery of scars. Thus, we state - scarring of wavefunctions on periodic orbits is a reminder of the classical fact that on its way, an arbitrary trajectory is shadowed by periodic orbits.

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[27] More precisely, trajectories having initial conditions in the two largest areas will perform the collision (backward in time) before returning to PSOS, whereas other areas are images of these largest areas by the Poincaré map (backward in time).

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Figure Captions

1 $\tilde{\lambda}^i(T) = \ln(|M(T)\cdot \mathbf{e}^i|)$, for $i = \|$ (continuous line), $\perp$ (dotted line) and r (dashed line), for $\beta = 2$ (see text for definition and initial conditions). As expected, $\tilde{\lambda}^{\|}(T)$ is bounded, whereas $\tilde{\lambda}^{\perp}(T)$ and $\tilde{\lambda}^{\mathrm{r}}(T)$ show linear behavior, emphasizing presence of hard chaos in the system.

2 Reduced PSOS defined by $q_2 = 0$, at different energies : $E_a = \epsilon_0 = 16$ (ground state energy, see Eqs. (6)), $E_b = \epsilon_1 = 25.5$ (see Eq. (10)), $E_c = \epsilon_N = 36$ (see Eq. (12)) and $E_d = 100$. The reduced dynamics is clearly chaotic. (See text for discussion about areas appearing in the PSOS).

3 Positions (top plot) and momenta (bottom plot) as function of time for an unstable periodic orbit of the system with $\beta = 2$ at classical energy $E = \epsilon_0$ (ground state energy). Continuous line: (q_1, p_1), dotted line (q_{-1}, p_{-1}) and dashed line (q_2, p_2). The four non-trivial eigenvalues of the monodromy matrix forms the quadruplet $(\Lambda, \Lambda^*, \Lambda^{-1}, \Lambda^{-1*})$, with $\Lambda = -367.64192 + \iota 64.633456$